Bounds for Functions of Multivariate Risks

Paul Embrechts [a], Giovanni Puccetti [b,1,*]

[a] *Department of Mathematics, ETH Zurich, CH-8092 Zurich, Switzerland*

[b] *Department of Mathematics for Decisions, University of Firenze, 50134 Firenze, Italy*

Abstract

Li, Scarsini, and Shaked [8] provide bounds on the distribution and on the tail for functions of dependent random vectors having fixed multivariate marginals. In this paper, we correct a result stated in the above article and we give improved bounds in the case of the sum of identically distributed random vectors. Moreover, we provide the dependence structures meeting the bounds when the fixed marginals are uniformly distributed on the k-dimensional hypercube. Finally, a definition of a multivariate risk measure is given along with actuarial/financial applications.

Key words: multivariate marginals, coupling, dual bounds, Value-at-Risk, risk measures
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1 Introduction

In this paper we provide bounds on the distribution and on the tail for functions of dependent risks having fixed multivariate marginals. Given a measurable function $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$ and k-variate random vectors $\mathbf{X}_1, \dots, \mathbf{X}_n$ on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, with associated distribution functions $F_1, \dots, F_n$, we investigate:

$$m_\psi(\mathbf{s}) := \inf\{\mathbb{P}[\psi(\mathbf{X}_1, \dots, \mathbf{X}_n) < \mathbf{s}] : \mathbf{X}_i \backsim F_i, 1 \leq i \leq n\}, \mathbf{s} \in \mathbb{R}^k, \quad (1)$$

$$M_\psi(\mathbf{s}) := \sup\{\mathbb{P}[\psi(\mathbf{X}_1, \dots, \mathbf{X}_n) \geq \mathbf{s}] : \mathbf{X}_i \backsim F_i, 1 \leq i \leq n\}, \mathbf{s} \in \mathbb{R}^k. \quad (2)$$

In the univariate case ($k = 1$) the above problems are equivalent and have received a considerable interest in the literature, see Embrechts and Puccetti [4] and references

* Corresponding author. Address: Dipartimento di Matematica per le Decisioni, via Lombroso 6/17, 50134 Firenze (FI), Italy. Tel. +39-055-4796824 Fax +39-055-4796800

Email addresses: embrechts@math.ethz.ch (Paul Embrechts), giovanni.puccetti@dmd.unifi.it (Giovanni Puccetti).

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therein. On the contrary, the multivariate-marginal set-up ($k > 1$), which constitutes a natural framework for risk management, has not been given much attention.

In fact, dealing with multivariate marginals causes extra problems. As shown in Scarsini [17], the concept of *copula* (see Nelsen [12, Def. 2.10.6]) as a tool to generate distribution functions from a set of marginals, becomes inadequate when dealing with the product of multivariate spaces. Compared to the univariate-marginal situation, this is a great disadvantage. Indeed, if $k = 1$ and $F_1, \ldots, F_n$ are continuous, then the set of n-dimensional copulas is isomorphic to the Fréchet class $\mathfrak{F}(F_1, \ldots, F_n)$ of distribution functions on $(\mathbb{R}^k)^n$ having such marginals. Moreover, Genest et al. [7, Prop. A] state that in the multivariate case the only copula generating a distribution function in $\mathfrak{F}(F_1, \ldots, F_n)$ for all possible choices of the F_i's is the independence measure $\Pi_{i=1}^n F_i$. This fact guarantees that the above problems at least make sense. The construction of different elements in $\mathfrak{F}(F_1, \ldots, F_n)$ has been treated in Cohen [2], Rüschendorf [15], Sánchez Algarra [16], Marco and Ruiz-Rivas [11], while an effort to create a copula-like tool in multivariate spaces has been made by Li et al. [9].

To our knowledge, Li et al. [8] seems to be the only paper where bounds on (1) and (2) are given. In the following, we correct a result given in the latter paper and give improved bounds on $m_\psi(\mathbf{s})$ and $M_\psi(\mathbf{s})$ for identically distributed risks. While sharpness of the bounds holds for any set of marginals only in the case of the sum of two random vectors, we derive an explicit solution for general portfolios of uniformly distributed risks.

Concerning applications in insurance and finance, we give a definition of *multivariate Value-at-Risk*.

2 Preliminaries and fundamental duality results

2.1 Notation

Given n (row) vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^k$, x_i^j indicates the j-th component of the i-th vector, for $i \in N := \{1, \ldots, n\}$ and $j \in K := \{1, \ldots, k\}$. Operations on and relations between vectors are defined componentwise, e.g. $\mathbf{x}_1 \leq (<) \, \mathbf{x}_2$ iff $x_1^j \leq (<) x_2^j$, for all $j \in K$. On the contrary, we write $\mathbf{x}_1 \nleq (\nless) \, \mathbf{x}_2$ when $x_1^{j'} > (\geq) \, x_2^{j'}$ for some $j' \in K$. Analogously, a k-valued real function f is *increasing* if for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^k$ with $\mathbf{x}_1 \leq \mathbf{x}_2$, we have $f(\mathbf{x}_1) \leq f(\mathbf{x}_2)$. Given a distribution function F, $L^1(F)$ denotes the class of all functions $f : \mathbb{R}^k \to \mathbb{R}$ which are F-integrable. For a vector $\mathbf{s} = (s^1, \ldots, s^k) \in \mathbb{R}^k$, we use also the notation $(-\infty, \mathbf{s}) := \prod_{j=1}^k (-\infty, s^j)$ and $[\mathbf{s}, +\infty) := \prod_{j=1}^k [s^j, +\infty)$. Finally, $\mathbb{I}$ stands for the unit interval on the real line and the indicator function of the set $B \subset \mathbb{R}^k$ is the function $1_B : \mathbb{R}^k \to \{0, 1\}$,

$$1_B(\mathbf{b}) := \begin{cases} 1 & \text{if } \mathbf{b} \in B, \\ 0 & \text{otherwise.} \end{cases}$$

For reason of notational simplicity, throughout the paper, we use the notation $\mathbf{x}$ both for vectors in $\mathbb{R}^k$ as well in $(\mathbb{R}^k)^n$; the appropriate meaning should always be clear from the context.

2.2 The Main Duality Theorem

On some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be $\mathbb{R}^k$-valued random vectors having given distribution functions $F_i(\mathbf{x}_i) = \mathbb{P}[X_i^j \le x_i^j, j \in K], i \in N$. Given k measurable functions $\psi_j : \mathbb{R}^n \to \mathbb{R}, j \in K$, we define the function $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$ as follows:

$$\psi(\mathbf{x}) = \psi(\mathbf{x}_1, \ldots, \mathbf{x}_n) := (\psi_1(x_1^1, \ldots, x_n^1), \ldots, \psi_k(x_1^k, \ldots, x_n^k)).$$

It will be useful to think about $\mathbf{X} := (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ as a portfolio of one-period multivariate insurance or financial risks. In this view, the function ψ makes sense if the risks $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are componentwise homogeneous.

Problems (1) and (2) have a dual counterpart, as stated in Ramachandran and Rüschendorf [13].

Theorem 1 (Main Duality Theorem) *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$, with $n > 1$, be random vectors on $\mathbb{R}^k$ having marginal distribution functions $F_1, \ldots, F_n$. Then*

$$\begin{aligned} m_\psi(\mathbf{s}) = \sup\Big\{ \sum_{i=1}^n \int_{\mathbb{R}^k} f_i dF_i : \; & f_i \in L^1(F_i), i \in N \textit{ with} \\ & \sum_{i=1}^n f_i(\mathbf{x}_i) \le 1_{(-\infty, \mathbf{s})}(\psi(\mathbf{x})) \textit{ for all } \mathbf{x} \in (\mathbb{R}^k)^n \Big\}, \end{aligned} \tag{3}$$

$$\begin{aligned} M_\psi(\mathbf{s}) = \inf\Big\{ \sum_{i=1}^n \int_{\mathbb{R}^k} f_i dF_i : \; & f_i \in L^1(F_i), i \in N \textit{ with} \\ & \sum_{i=1}^n f_i(\mathbf{x}_i) \ge 1_{[\mathbf{s}, +\infty)}(\psi(\mathbf{x})) \textit{ for all } \mathbf{x} \in (\mathbb{R}^k)^n \Big\}. \end{aligned} \tag{4}$$

According to Lindvall [10, (1.1)], we call every random vector $\mathbf{X}^C = (\mathbf{X}_1^C, \ldots, \mathbf{X}_n^C)$ with df in $\mathfrak{F}(F_1, \ldots, F_n)$ a *coupling*. Given a coupling $\mathbf{X}^C$ and two sets of functions $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)$ and $\hat{\mathbf{g}} = (\hat{g}_1, \ldots, \hat{g}_n)$ which are admissible for (3), respectively for (4), we obviously have that

$$\mathbb{P}[\psi(\mathbf{X^C}) < \mathbf{s})] \geq m_\psi(\mathbf{s}) \geq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{f}_i dF_i, \tag{5}$$

$$\mathbb{P}[\psi(\mathbf{X^C}) \geq \mathbf{s})] \leq M_\psi(\mathbf{s}) \leq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{g}_i dF_i. \tag{6}$$

In this case we call $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ *dual choices* for (3), respectively for (4). A coupling and a dual choice which satisfy (5) ((6)) with two equalities will be called an *optimal coupling* and a *dual solution*, respectively, since they solve problem (1) ((2)).

In the case of identically distributed random vectors, Remark 2 in Gaffke and Rüschendorf [6] can be easily adapted to give the following corollary.

Corollary 2 (Reduced Duality) *Under the assumptions of Theorem 1, let $F_i = F, i \in N$ and $\psi_j, j \in K$ be symmetric, i.e. $\psi_j(x_1, \dots, x_n) = \psi_j(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for all permutations $\sigma : N \to N$. Then*

$$\begin{aligned} m_\psi(\mathbf{s}) = \sup\Big\{ n \int_{\mathbb{R}^k} f dF : f \in L^1(F) \textit{ with} \\ \sum_{i=1}^{n} f(\mathbf{x}_i) \leq 1_{(-\infty,\mathbf{s})}(\psi(\mathbf{x})) \textit{ for all } \mathbf{x} \in \times_{i=1}^{n} \mathrm{supp}(F) \Big\}, \end{aligned} \tag{7}$$

$$\begin{aligned} M_\psi(\mathbf{s}) = \inf\Big\{ n \int_{\mathbb{R}^k} f dF : f \in L^1(F) \textit{ with} \\ \sum_{i=1}^{n} f(\mathbf{x}_i) \geq 1_{[\mathbf{s},+\infty)}(\psi(\mathbf{x})) \textit{ for all } \mathbf{x} \in \times_{i=1}^{n} \mathrm{supp}(F) \Big\}. \end{aligned} \tag{8}$$

The dual formulations (3) and (4) are very difficult to solve. For increasing functionals ψ, solutions under general marginal distributions are known only when $k = 1$ and $n = 2$; see Embrechts and Puccetti [4]. For $\psi = +$, the sum operator, Li et al. [8] give $m_\psi(\mathbf{s})$ for $n = 2$ and arbitrary k. Finally, when $n > 2$, the only explicit solution known is given in Rüschendorf [14] for the sum of risks uniformly distributed on the unit interval.

3 Standard bounds

In line with Embrechts and Puccetti [4], we call *standard bounds* those bounds obtained by choosing *piecewise-constant* dual choices in (3) and (4).

Theorem 3 *Let $\mathbf{X}_1, \dots, \mathbf{X}_n$, $n > 1$, be random vectors on $\mathbb{R}^k$ having marginals $F_1, \dots, F_n$. Let $\psi_1, \dots, \psi_k : \mathbb{R}^n \to \mathbb{R}$ be increasing in each coordinate and strictly* *increasing in the last. Then, for every* $\mathbf{s} \in \mathbb{R}^k$*, we have*

$$m_\psi(\mathbf{s}) \geq \sup_{\substack{\mathbf{u}\in(\mathbb{R}^k)^n,\\ \psi(\mathbf{u})\leq \mathbf{s}}} \left[\sum_{i=1}^{n-1} F_i(\mathbf{u}_i) + F_n^-(\mathbf{u}_n) - n + 1\right]^+, \tag{9}$$

where $F_n^-(\mathbf{u}_n) := \mathbb{P}[X_n^j < u_n^j, j \in K]$.

Proof Fix $\mathbf{u} \in (\mathbb{R}^k)^n$ with $\psi(\mathbf{u}) \leq \mathbf{s}$ and define the functions $\hat{f}_1^{\mathbf{u}}, \ldots, \hat{f}_n^{\mathbf{u}}$,

$$\hat{f}_i^{\mathbf{u}}(\mathbf{x}) : = \begin{cases} 1/n & \text{if } \mathbf{x} \leq \mathbf{u}_i, \\ 1/n - 1 & \text{otherwise} \end{cases}, i = 1, \ldots, n-1,$$
$$\hat{f}_n^{\mathbf{u}}(\mathbf{x}) : = \begin{cases} 1/n & \text{if } \mathbf{x} < \mathbf{u}_n, \\ 1/n - 1 & \text{otherwise.} \end{cases}$$

We show that $\hat{\mathbf{f}}^{\mathbf{u}} := (\hat{f}_1^{\mathbf{u}}, \ldots, \hat{f}_n^{\mathbf{u}})$ is a dual choice for (3). Since $\sum_{i=1}^n \hat{f}_i^{\mathbf{u}} \leq \sum_{i=1}^n 1/n = 1$, for admissibility it is sufficient to show that $\sum_{i=1}^n \hat{f}_i^{\mathbf{u}}(\mathbf{x}_i) \leq 0$ for every $\mathbf{x} \in (\mathbb{R}^k)^n$ such that $\psi(\mathbf{x}) \nleq \mathbf{s}$. For this, suppose that for some $\tilde{\mathbf{x}}$, $\sum_{i=1}^n \hat{f}_i^{\mathbf{u}}(\tilde{\mathbf{x}}_i) > 0$. By definition of the $\hat{f}_i^{\mathbf{u}}$'s, this implies that $\hat{f}_i^{\mathbf{u}}(\tilde{\mathbf{x}}_i) = 1/n$ for every $i \in N$, yielding $\tilde{\mathbf{x}}_i \leq \mathbf{u}_i, i = 1, \ldots n-1$ and $\tilde{\mathbf{x}}_n < \mathbf{u}_n$. Since the ψ_i's are increasing and strictly increasing in the last coordinate, we have

$$\begin{aligned}\psi(\tilde{\mathbf{x}}) &= (\psi_1(\tilde{x}_1^1, \ldots, \tilde{x}_n^1), \ldots, \psi_k(\tilde{x}_1^k, \ldots, \tilde{x}_n^k)) \\ &< (\psi_1(u_1^1, \ldots, u_n^1), \ldots, \psi_k(u_1^k, \ldots, u_n^k)) = \psi(\mathbf{u}) \leq \mathbf{s},\end{aligned}$$

which proves admissibility of $\hat{\mathbf{f}}^{\mathbf{u}}$. Substituting the $\hat{f}_i^{\mathbf{u}}$'s in (3) we find

$$\begin{aligned} m_\psi(\mathbf{s}) &\geq 1/n\left[\sum_{i=1}^{n-1}(F_i(\mathbf{u}_i) + (1-n)(1 - F_i(\mathbf{u}_i))) + F_n^-(\mathbf{u}_n) + (1-n)(1 - F_n^-(\mathbf{u}_n))\right] \\ &= \sum_{i=1}^{n-1} F_i(\mathbf{u}_i) + F_n^-(\mathbf{u}_n) - n + 1. \end{aligned}$$

Noting that $m_\psi(\mathbf{s})$ is non-negative and taking the supremum over all $\mathbf{u} \in (\mathbb{R}^k)^n$ such that $\psi(\mathbf{u}) \leq \mathbf{s}$, we get (9). □

We give an analogous bound for $M_\psi(\mathbf{s})$.

Theorem 4 *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_n$*,* $n > 1$*, be random vectors on* $\mathbb{R}^k$ *having continuous marginals* $F_1, \ldots, F_n$*. Let* $\psi_1, \ldots, \psi_k : \mathbb{R}^n \to \mathbb{R}$ *be strictly increasing in each coor-*

dinate. Then, for every $\mathbf{s} \in \mathbb{R}^k$*, we have*

$$M_\psi(\mathbf{s}) \leq \inf_{\substack{\mathbf{u}\in(\mathbb{R}^k)^n,\\ \psi(\mathbf{u})\leq \mathbf{s}}} \quad \min\left\{1/2\left[n + \sum_{i=1}^{n}(\overline{F}_i(\mathbf{u}_i) - F_i(\mathbf{u}_i))\right], 1\right\}, \tag{10}$$

where $\overline{F}_i(\mathbf{u}_i) := \mathbb{P}[X_i^j \geq u_i^j, j \in K], i \in N$.

Proof The proof is analogous to that of Theorem 3, with the dual choice $\hat{\mathbf{f}}^{\mathbf{u}}$ replaced by

$$\hat{f}_i^{\mathbf{u}}(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \leq \mathbf{u}_i, \mathbf{x} \neq \mathbf{u}_i, \\ 1 & \text{if } \mathbf{x} \geq \mathbf{u}_i, \\ 1/2 & \text{otherwise.} \end{cases}, i \in N.$$

Remark 5 *The bound given by this theorem can be adapted to non-continuous marginals by adding* $(1/2)\sum_{i=1}^n \mathbb{P}[\mathbf{X}_i = \mathbf{u}_i]$ *to the first argument of the* min *operator in* (10)*.*

For general ψ_i's, (9) and (10) are difficult to calculate. In the case of the sum of vectors they reduce to easier expressions, as the following example shows.

Example 6 *In case of* $\psi_j = +$, $j \in K$*, we obtain*

$$m_+(\mathbf{s}) \geq \sup_{\mathbf{u}_1,\dots,\mathbf{u}_{n-1}\in\mathbb{R}^k} \left[\sum_{i=1}^{n-1} F_i(\mathbf{u}_i) + F_n^-\left(\mathbf{s} - \sum_{i=1}^{n-1}\mathbf{u}_i\right) - n + 1\right]^+, \tag{11}$$

$$\begin{aligned} M_+(\mathbf{s}) \leq \inf_{\mathbf{u}_1,\dots,\mathbf{u}_{n-1}\in\mathbb{R}^k} \min\Bigg\{1/2\Bigg[n + \sum_{i=1}^{n-1}(\overline{F}_i(\mathbf{u}_i) - F_i(\mathbf{u}_i)) \\ + \overline{F}_n\left(\mathbf{s} - \sum_{i=1}^{n-1}\mathbf{u}_i\right) - F_n\left(\mathbf{s} - \sum_{i=1}^{n-1}\mathbf{u}_i\right)\Bigg], 1\Bigg\}. \end{aligned} \tag{12}$$

When $n = 2$, (11) improves the left-hand side of (2.5) in Li et al. [8]. Note that in that paper distribution functions are defined to be continuous from below. Moreover, (12) is the correct version of the right-hand side of the last line of Theorem 2.2 in the above reference. In fact, as the following counterexample shows, the latter is not correct.

Example 7 *Let* $\mathbf{X}_1, \mathbf{X}_2$ *be bivariate random vectors uniformly distributed on the unit square, i.e.* $\mathbf{X}_i \backsim U(\mathbb{I}^2), i = 1, 2$*. For* $\mathbf{s} = (1, 1)$*, the last line of Theorem 2.2* *in Li et al. [8] gives*

$$\sup_{\mathfrak{F}(U(\mathbb{I}^2),U(\mathbb{I}^2))} \mathbb{P}[\mathbf{X}_1 + \mathbf{X}_2 \geq (1,1)] = \inf_{\mathbf{u}+\mathbf{v}=(1,1)} \min\{\mathbb{P}[\mathbf{X}_1 \geq \mathbf{u}] + \mathbb{P}[\mathbf{X}_2 \geq \mathbf{v}], 1\}$$
$$\leq \mathbb{P}[\mathbf{X}_1 \geq (1,0)] + \mathbb{P}[\mathbf{X}_2 \geq (0,1)] = 0.$$

This is wrong since it is possible to set $\mathbf{X}_2^C = (1,1) - \mathbf{X}_1$ *to obtain* $\mathbb{P}[\mathbf{X}_1 + \mathbf{X}_2^C \geq (1,1)] = 1$*. It is not difficult to show that* (12) *provides the correct value in this case.*

In the univariate case, the bounds stated in Theorems 3 and 4 are equivalent and pointwise best-possible when $n = 2$; see Rüschendorf [14]. The corresponding optimal coupling is given in Frank et al. [5].

In the multivariate set-up, the situation is different. Theorem 3.3 in Li et al. [8] states sharpness of (11) for the sum of two k-variate risks. In the proof of this theorem, which is based on Strassen [18, Th. 11], the authors do not actually use any continuity assumptions on the distribution function of $(\mathbf{X}_1 + \mathbf{X}_2)$ and their result holds for general sets of marginals. Note that in equation (3.3) in the above paper the last component inside the supremum should be $P_1((-\infty, \mathbf{t} - \mathbf{a}]^c)$; see also (5) in Rüschendorf [14]. The bound (12), though being the best-possible standard bound, behaves differently. We show in Section 6 that the latter is not sharp even when $n = k = 2$. We also remark that Theorem 11 in Strassen [18] cannot be applied in this case.

4 Uniform multivariate marginals

In this section, we provide optimal couplings solving problems (1) and (2) in the case of the sum of random vectors uniformly distributed on $\mathbb{I}^k$. The following theorem explores the case of the sum of two vectors.

Theorem 8 *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be random vectors uniformly distributed on* $\mathbb{I}^k$ *and* $\mathbf{s} \in [\mathbf{0}, +\infty)$*. Then*

$$m_+(\mathbf{s}) = \prod_{j=1}^{k} \hat{s}^j, \tag{13}$$

$$M_+(\mathbf{s}) = \prod_{j=1}^{k} (1 - \hat{s}^j), \tag{14}$$

where $\hat{s}^j := \min\{[s^j - 1]^+, 1\}$, $j \in K$.

Proof First, note that the coupling defined in Example 7 yields $m_+(\mathbf{1}) = 0$, where $\mathbf{1} := (1, 1)$. Since we trivially have $m_+(2\mathbf{1}) = 1$, it suffices to consider $\mathbf{s} \in [1, 2]^k$.

With respect to (13), take $\mathbf{X}_1^C \backsim U(\mathbb{I}^k)$ and let $\mathbf{X}_2^C = F(\mathbf{X}_1^C)$, where the function $F : \mathbb{I}^k \to \mathbb{I}^k$ is defined as follows:

$$F(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{if } \mathbf{x} < \hat{\mathbf{s}}, \\ 1 + \hat{\mathbf{s}} - \mathbf{x} & \text{otherwise.} \end{cases}$$

Note that $\mathbf{X}_2^C$ has univariate marginals uniformly distributed on $\mathbb{I}$. Moreover, for $j_1 \neq j_2$, the random variables $X_2^{Cj_1}$ and $X_2^{Cj_2}$ depend only on $X_1^{Cj_1}$ and on $X_1^{Cj_2}$, respectively. Since the latter are independent, the vector $\mathbf{X}_2^C$ is uniformly distributed on $\mathbb{I}^k$. For every $j \in K$, we have that

$$X_1^{Cj} + X_2^{Cj} = \begin{cases} 2X_1^{Cj} < 2\hat{s}^j \leq s^j & \text{if } X_1^{Cj} < \hat{s}^j, \\ 1 + \hat{s}^j = s^j & \text{otherwise.} \end{cases}$$

Hence $m_+(\mathbf{s}) \leq \mathbb{P}[\mathbf{X}_1^C + \mathbf{X}_2^C < \mathbf{s}] = \prod_{i=1}^n \hat{s}^j$. To prove the converse inequality, we show that the function $f : \mathbb{R}^k \to \mathbb{R}$, $f(\mathbf{x}) := (1/2)1_{(-\infty,\hat{\mathbf{s}})}(\mathbf{x})$ is an admissible choice for (7). Since $2f \leq 1$, it is sufficient to fix arbitrary vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{I}^k$ and check that $f(\mathbf{x}_1) + f(\mathbf{x}_2) > 0$ implies $\mathbf{x}_1 + \mathbf{x}_2 < \mathbf{s}$. Under such an hypothesis, it is necessary that at least $f(\mathbf{x}_1) = 1/2$, say. It follows that $\mathbf{x_1} < \hat{\mathbf{s}}$, implying $\mathbf{x}_1 + \mathbf{x}_2 < \hat{\mathbf{s}} + \mathbf{1} = \mathbf{s}$. Hence, f is admissible in (7) and $m_+(\mathbf{s}) \geq 2 \int_{\mathbb{I}^k} f dU(\mathbb{I}^k) = \prod_{j=1}^k \hat{s}^j$. The proof for (14) follows analogously by choosing the same coupling and the dual choice $f : \mathbb{R}^k \to \mathbb{R}$, $f(\mathbf{x}) := (1/2)1_{[\hat{\mathbf{s}},+\infty)}(\mathbf{x})$. □

Remark 9 *The first part of the above proof is not necessary since* (13) *is implied by Li et al. [8, Th. 3.3]. However, our* coupling-dual *approach avoids complicated multivariate optimizations.*

The following theorem, which we prove in Appendix A, provides an optimal coupling of more than two risks, hence extending Rüschendorf [14, Th. 1] to the multivariate set-up.

Theorem 10 *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_{k+1}$ *be random vectors uniformly distributed on* $\mathbb{I}^k$ *and* $\mathbf{s} \in [k, k+1]^k$. *Then*

$$M_+(\mathbf{s}) = \frac{\prod_{j=1}^k (k + 1 - s^j)}{k!}. \tag{15}$$

Remark 11 *Figure 2, right, to be found in Section 5, illustrates* (15) *for* $k = 2$. *It is important to point out the following remarks.*

(i) *The optimal coupling for the sum of three vectors uniformly distributed on the unit square, which is illustrated in Figure 1, is defined by*

$$\mathbf{X}_1^C = \mathbf{X}_1, \mathbf{X}_2^C = F(\mathbf{X}_1^C) \text{ and } \mathbf{X}_3^C = F \circ F(\mathbf{X}_1^C),$$

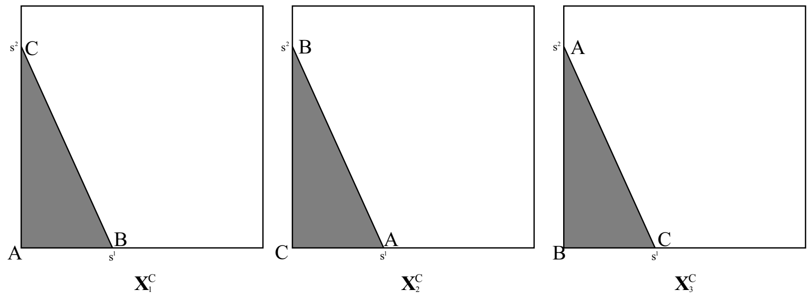

Fig. 1. Optimal coupling in Theorem 10 when $k = 2$.

where $F : \mathbb{I}^2 \to \mathbb{I}^2$,

$$F(\mathbf{x}) := \begin{cases} \left(-x^1 - \frac{s^1}{s^2}x^2 + s^1, \frac{s^2}{s^1}x^1\right) & \textit{if } \mathbf{x} \in A_2, \\ \mathbf{x} & \textit{otherwise,} \end{cases}$$

with $A_2 := \{\mathbf{x} \in \mathbb{I}^2 : \sum_{j=1}^{2} \frac{x^j}{s^j} \le 1\}$.

(ii) *In the proof of this theorem, which we give in Appendix A, we show that an upper bound on* $M_+(\mathbf{s})$ *is available for all* $\mathbf{s} \in [\mathbf{0}, +\infty), k \ge 2$ *and* $n \ge 2$. *Unfortunately, it seems difficult to provide optimal couplings in general dimensions.*

(iii) *It seems difficult to find* $m_+(\mathbf{s})$ *for the sum of more than two random vectors even under the uniform-marginal assumption. A lower bound on the latter value will be computed using Theorem 14 below.*

(iv) *Note that the optimal coupling defined in the proof of Theorem 8 is simply the product of the optimal univariate couplings given in Rüschendorf [14, (10)]. Unfortunately, the same technique does not work for multivariate vectors (i.e* $k \ge 2$*) when* $n \ge 3$.

5 Non-negative, identically distributed risks

When $n > 2$ and the fixed marginal distributions are not uniform, it is difficult to find $m_+(\mathbf{s})$ and $M_+(\mathbf{s})$. In Section 3, we used piecewise-constant functions as admissible choices to produce so-called standard bounds. If we restrict to the case of the sum of non-negative identically distributed risks, it is possible to find *piecewise-linear* choices yielding improved bounds. Recall from Corollary 2 that if $\hat{f}, \hat{g}$ are dual choices for (7) and (8), respectively, then we have

$$m_+(\mathbf{s}) \ge n \int_{\mathbb{R}^k} \hat{f} dF, \tag{16}$$

$$M_+(\mathbf{s}) \le n \int_{\mathbb{R}^k} \hat{g} dF. \tag{17}$$

Theorem 12 *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_n$, $n > 1$, *be random vectors on* $\mathbb{R}^k$ *identically distributed as* F, *a non-negative, continuous distribution function. Then, for every*

$\mathbf{s} \in [\mathbf{0}, +\infty)$, *we have*

$$m_+(\mathbf{s}) \geq n \sup_{\boldsymbol{\gamma} \in [0, \frac{1}{n}\mathbf{s})} \int_{[\mathbf{0}, +\infty)} f^*_{\boldsymbol{\gamma}}(\mathbf{x}) dF(\mathbf{x}), \tag{18}$$

where

$$f^*_{\boldsymbol{\gamma}}(\mathbf{x}) := 1/n - \min \left\{ \max_{j \in K} \frac{[x^j - \gamma^j]^+}{s^j - n\gamma^j}, 1 \right\},$$

for fixed $\boldsymbol{\gamma} = (\gamma^1, \ldots, \gamma^k) \in [0, \frac{1}{n}\mathbf{s})$.

Proof By (16) and considerations above, we have to show that the F-integrable function $f^*_{\boldsymbol{\gamma}}$ is admissible for problem (7), i.e. that for every $\mathbf{x} \in \times_{i=1}^n [\mathbf{0}, +\infty)$ we have that $\sum_{i=1}^n f^*_{\boldsymbol{\gamma}}(\mathbf{x}_i) \leq 1_{(-\infty, \mathbf{s})}(\sum_{i=1}^n \mathbf{x}_i)$. Since $\sum_{i=1}^n f^*_{\boldsymbol{\gamma}} \leq n(1/n) = 1$, we fix $\mathbf{x}$ such that $\sum_{i=1}^n \mathbf{x}_i \not< \mathbf{s}$ and show that $\sum_{i=1}^n f^*_{\boldsymbol{\gamma}}(\mathbf{x}_i) \leq 0$. If $f^*_{\boldsymbol{\gamma}}(\mathbf{x}_{\hat{\imath}}) = 1/n - 1$ for some $\hat{\imath} \in N$, then $\sum_{i=1}^n f^*_{\boldsymbol{\gamma}}(\mathbf{x}_i) = f^*_{\boldsymbol{\gamma}}(\mathbf{x}_{\hat{\imath}}) + \sum_{i \neq \hat{\imath}} f^*_{\boldsymbol{\gamma}}(\mathbf{x}_i) \leq 1/n - 1 + (n-1)/n = 0$. Hence we can restrict to $\mathbf{x}_i \in \prod_{j=1}^k [0, s^j - (n-1)\gamma^j), i \in N$ with $\sum_{i=1}^n \mathbf{x}_i^{\hat{\jmath}} \geq s^{\hat{\jmath}}$ for some $\hat{\jmath} \in K$. Define the sets $\overline{I} := \{i \in N : x_i^j \leq \gamma^j, j \in K\}$ and $I := N \setminus \overline{I}$ and note that

$$\sum_{i=1}^n x_i^{\hat{\jmath}} = \sum_{i \in \overline{I}} x_i^{\hat{\jmath}} + \sum_{i \in I} x_i^{\hat{\jmath}} \geq s^{\hat{\jmath}}.$$

Since $x_i^{\hat{\jmath}} \leq \gamma^{\hat{\jmath}}$ when $i \in \overline{I}$, we have that

$$\sum_{i \in I} x_i^{\hat{\jmath}} \geq s^{\hat{\jmath}} - \left|\overline{I}\right| \gamma^{\hat{\jmath}}.$$

Finally, we can write

$$\begin{aligned}
\sum_{i=1}^n f^*_{\boldsymbol{\gamma}}(\mathbf{x}_i) &= n(1/n) - \sum_{i \in I} \max_{j \in K} \frac{x_i^j - \gamma^j}{s^j - n\gamma^j} \leq 1 - \sum_{i \in I} \frac{x_i^{\hat{\jmath}} - \gamma^{\hat{\jmath}}}{s^{\hat{\jmath}} - n\gamma^{\hat{\jmath}}} \\
&= 1 - \frac{(\sum_{i \in I} x_i^{\hat{\jmath}} - |I| \gamma^{\hat{\jmath}})}{s^{\hat{\jmath}} - n\gamma^{\hat{\jmath}}} \leq 1 - \frac{(s^{\hat{\jmath}} - \left|\overline{I}\right| \gamma^{\hat{\jmath}} - |I| \gamma^{\hat{\jmath}})}{s^{\hat{\jmath}} - n\gamma^{\hat{\jmath}}} = 0.
\end{aligned}$$

The theorem follows from arbitrariness of $\mathbf{x} \in \times_{i=1}^n [\mathbf{0}, +\infty)$. □

Remark 13 *There are several points worth noting regarding this theorem.*

(i) *For* $\boldsymbol{\gamma}$ *tending to* $(1/n)\mathbf{s}$, *and for the distributions of actuarial/financial interest used in Section 6,* $f^*_{\boldsymbol{\gamma}}$ *converges in the sup-norm to an admissible choice yielding the standard bound* (11). *Consequently, the dual bound* (18) *is always better (≥) than* (11). *In Section 6 we will show that for such distributions it is actually strictly better (>).*

(ii) *If* $F = \min\{G, \ldots, G\}$ *for a univariate, continuous, non-negative distribution function* G*, the support of* F *is the set* $\{\mathbf{x} \in \mathbb{R}^k : x^1 = \cdots = x^k\}$*; see (27) in Dhaene et al. [3]. In this special case, for* $s^j = s^o \geq 0$, $j \in K$, (18) *reduces to*

$$m_+(\mathbf{s}) \geq n \sup_{\gamma^o \in [0, \frac{s^o}{n})} \int_0^{+\infty} f^*_{\gamma^o}(x) dG(x), \tag{19}$$

where

$$f^*_{\gamma^o}(x) = \begin{cases} 1/n - \frac{[x-\gamma^o]^+}{s^o - n\gamma^o} & \text{if } x \in [0, s^o - (n-1)\gamma^o), \\ 1/n - 1 & \text{otherwise.} \end{cases}$$

It is easy to check that (19) *corresponds to (4.4) in Embrechts and Puccetti [4]. In fact, under the distribution function* $F = \min\{G, \ldots, G\}$ *we have that* $\mathbb{P}[X_i^j = X_i^1, j \in K] = 1, i \in N$*, implying* $\mathbb{P}\left[\sum_{i=1}^n \mathbf{X}_i < (s^o, \ldots, s^o)\right] = \mathbb{P}\left[\sum_{i=1}^n X_i^1 < s^o\right]$*, which is a univariate problem. Of course, it is also possible to find* (19) *by setting* $k = 1$ *and* $s^1 = s^o$*. To this extent, Theorem 12 extends Embrechts and Puccetti [4, Th. 4.2].*

Theorem 12 can be used to compute a lower bound on $m_+(\mathbf{s})$ in the case of uniform marginals. The results of the optimizations are shown in Figure 2, left. Our next theorem gives an upper bound on $M_\psi(\mathbf{s})$.

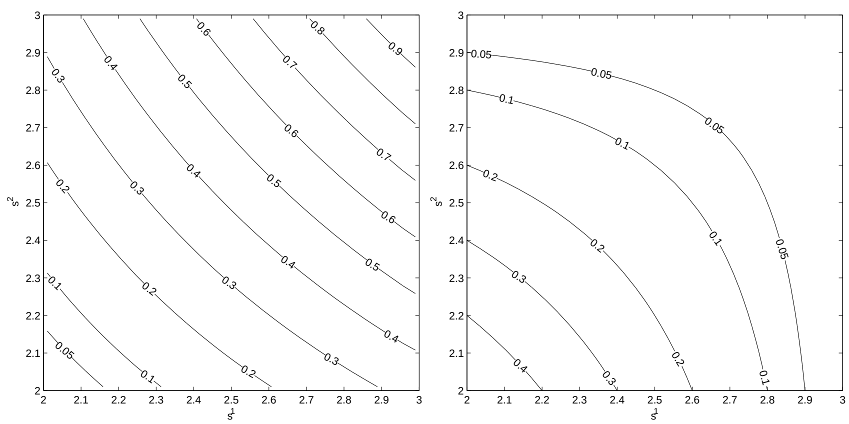

Fig. 2. Level sets for the dual bound (18) on $m_+(s^1, s^2)$ (left) and for the function $M_+(s^1, s^2)$ (right) for three random vectors uniformly distributed on the unit square.

Theorem 14 *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_n$, $n > 1$*, be random vectors on* $\mathbb{R}^k$ *identically distributed as* F*, a non-negative, continuous distribution function. Then, for every* $\mathbf{s} \in [\mathbf{0}, +\infty)$*, we have*

$$M_+(\mathbf{s}) \leq n \inf_{\gamma \in [0, \frac{1}{n}\mathbf{s})} \int_{[\mathbf{0}, +\infty)} f^*_\gamma(\mathbf{x}) dF(\mathbf{x}), \tag{20}$$

where

$$f^*_{\gamma}(\mathbf{x}) := \begin{cases} \frac{\left[\sum_{j=1}^k x^j - \gamma\right]^+}{s - n\gamma} & \text{if } \mathbf{x} \in \prod_{j=1}^k [0, \frac{s^j - \gamma^j}{n-1}), \\ \frac{1}{2} + \frac{1}{2} 1_{[\boldsymbol{\gamma}, +\infty)}(\mathbf{x}) & \text{otherwise,} \end{cases}$$

for fixed $\boldsymbol{\gamma} = (\gamma^1, \dots, \gamma^k) \in [0, \frac{1}{n}\mathbf{s})$, *with* $\gamma := \sum_{j=1}^k \gamma^j$ *and* $s := \sum_{j=1}^k s^j$.

Proof By (17) and considerations above, we have to show that the F-integrable function f^*_{γ} is admissible for (8), i.e. that for every $\mathbf{x} \in \times_{i=1}^n [\mathbf{0}, +\infty)$ we have that $\sum_{i=1}^n f^*_{\gamma}(\mathbf{x}_i) \geq 1_{[\mathbf{s}, +\infty)}(\sum_{i=1}^n \mathbf{x}_i)$. Since f^*_{γ} is non-negative, we fix $\mathbf{x}$ with $\sum_{i=1}^n \mathbf{x}_i \geq \mathbf{s}$ and show that $\sum_{i=1}^n f^*_{\gamma}(\mathbf{x}_i) \geq 1$. It will be useful to divide the proof in two steps.

Step 1: Suppose that $\mathbf{x}_i \in \prod_{j=1}^k \left[0, \frac{s^j - \gamma^j}{n-1}\right), i \in N$ and define the sets $\overline{I} := \{i : \sum_{j=1}^k x_i^j \leq \gamma\}$, $I := N \setminus \overline{I}$. Then, we have

$$s \leq \sum_{j=1}^k \sum_{i=1}^n x_i^j = \sum_{i=1}^n \sum_{j=1}^k x_i^j = \sum_{i \in I} \sum_{j=1}^k x_i^j + \sum_{i \in \overline{I}} \sum_{j=1}^k x_i^j,$$

which, by definition of $\overline{I}$, leads to

$$\sum_{i \in I} \sum_{j=1}^k x_i^j \geq s - \sum_{i \in \overline{I}} \sum_{j=1}^k x_i^j \geq s - \left|\overline{I}\right| \gamma.$$

Hence, we can write

$$\sum_{i=1}^n f^*_{\gamma}(\mathbf{x}_i) = \sum_{i=1}^n \frac{\left[\sum_{j=1}^k x_i^j - \gamma\right]^+}{s - n\gamma} = \frac{\sum_{i \in I} \sum_{j=1}^k x_i^j - |I| \gamma}{s - n\gamma}$$
$$\geq \frac{s - \left|\overline{I}\right| \gamma - |I| \gamma}{s - n\gamma} = 1.$$

Step 2: Suppose that $x_i^j \geq \frac{s^j - \gamma^j}{n-1} \geq \gamma^j$ for some $i \in N$ and $j \in K$. Assume also that $x_i^{j'} < \gamma^{j'}$ for some $j' \neq j$. In this case $f(\mathbf{x}_i) = 1/2$. If $\mathbf{x}_{i'}$ does not lie in $\prod_{j=1}^k [0, \frac{s^j - \gamma^j}{n-1})$ for some $i' \neq i$, then $\sum_{i=1}^n f^*_{\gamma}(\mathbf{x}_i) \geq 1/2 + 1/2 = 1$. If, instead, $\mathbf{x}_{i'} \in \prod_{j=1}^k [0, \frac{s^j - \gamma^j}{n-1})$ for all $i' \neq i$, we have

$$\sum_{i=1}^n x_i^{j'} = \sum_{i' \neq i} x_{i'}^{j'} + x_i^{j'} < \sum_{i' \neq i} \frac{s^{j'} - \gamma^{j'}}{n-1} + \gamma^{j'} = s^{j'},$$

which is contrary to our assumption. Finally, consider the case where there exists $i \in N$ such that $x_i^j \geq \frac{s^j - \gamma^j}{n-1}$ for some $j \in K$ with $x_i^{j'} \geq \gamma^{j'}$ for all $j' \neq j$. In this particular case $\sum_{i=1}^n f^*_{\gamma}(\mathbf{x}_i) \geq f^*_{\gamma}(\mathbf{x}_i) = 1$.

Admissibility of f^*_γ follows from the arbitrariness of $\mathbf{x} \in \times_{i=1}^n [\mathbf{0}, +\infty)$. $\square$

Remark 15 *Remark 13,* (i) *holds analogously for this theorem. For $n = 2$ the function $f^*_\gamma \left(1 - 1_{\Pi_{j=1}^k [0, s^j - \gamma^j)} + 1_{\Pi_{j=1}^k [\gamma^j, s^j - \gamma^j)}\right)$ improves the function f^*_γ. We consider this in Figure 4.*

In the univariate-marginal case there is a *natural* choice of the linear function yielding the so-called *dual bound*; see Embrechts and Puccetti [4, Th. 4.2]. In the multivariate setting, instead, that choice is not straightforward. Of course, if f and g are two dual choices for (7) (resp. (8)) with $f \geq (\leq)\, g$, then f will provide a better lower (upper) bound on $m_+(\mathbf{s})$ ($M_+(\mathbf{s})$) for all possible sets of fixed marginals and non-negative vectors $\mathbf{s}$. On the contrary, if f and g are not ordered in such a way, then it is possible to find a distribution function G for which either g provides a better bound than f or viceversa. For instance, consider the following function:

$$g^*_\gamma(\mathbf{x}) := \min \left\{ \frac{\left[\sum_{j=1}^k x^j - \gamma\right]^+}{s - n\gamma}, 1 \right\}. \tag{21}$$

It follows easily from Step 1, that g^*_γ is a dual choice for (8). Since g^*_γ does not include any standard dual choice as a particular case, it may fail to improve the corresponding bound (12). However, it turns out that g^*_γ yields a bound which is better than (20) in many cases of interest. Note also that Remark 13, (ii) holds for g^*_γ. Whenever several dual choices are available, an overall better bound is produced by taking the pointwise minimum/maximum among the corresponding bounds. We will follow this methodology in Section 6. An end-user working with some particular fixed marginal distribution functions may find it useful to construct an *ad-hoc* admissible choice yielding a very good bound within the specific context.

6 Applications

In this section, we illustrate the bounds provided by Theorems 12 and 14 within a financial/insurance risk management context. Random vectors will be referred to as portfolios, the individual random sub-vectors as risks. We consider portfolios of identically distributed, non-negative risks. As fixed marginals, we choose two bivariate distribution functions of actuarial and financial interest. The first one is the *bivariate Pareto*, whose tail function $\overline{F}_\theta, \theta > 0$ is defined in Nelsen [12, Ex. 2.14]. The second one, which we call *bivariate Log-Normal*, is the product of two univariate Log-Normal distribution functions with parameters (μ, σ^2). In the following, except as stated otherwise, we take $\theta = 0.9, \mu = -0.2$ and $\sigma^2 = 1$.

In Figure 3, we give standard and dual bounds on $\mathbb{P}[\sum_{i=1}^n \mathbf{X}_i < (s, s)]$ for portfolios consisting of two and three bivariate risks. We stress that the standard bound (11)

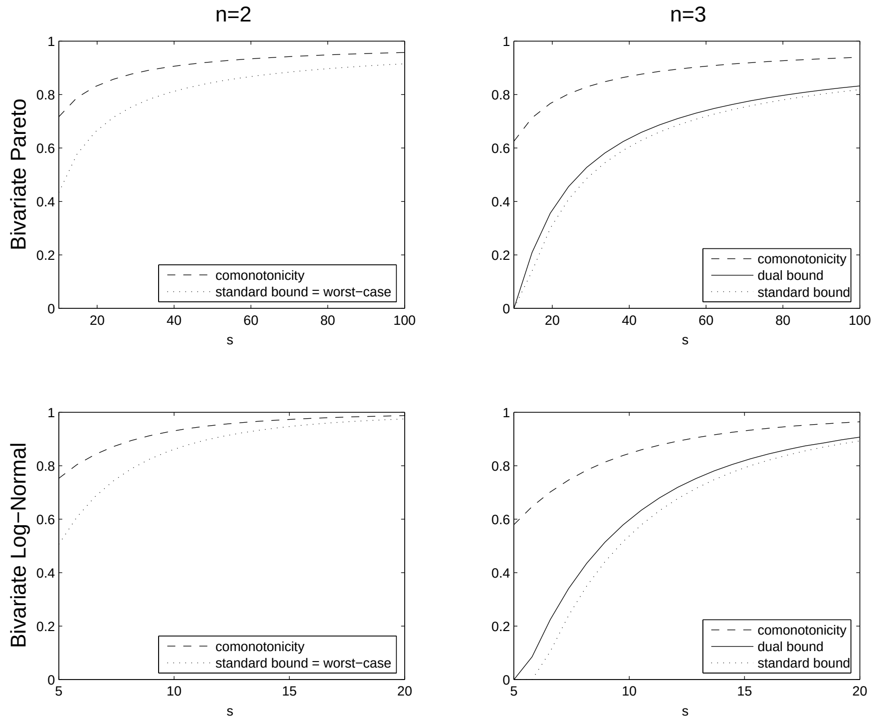

Fig. 3. Range for $\mathbb{P}[\sum_{i=1}^{n} \mathbf{X}_i < (s, s)]$ for two and three risks identically distributed as a bivariate Pareto or bivariate Log-Normal distribution function. Together with the comonotonic situation, we represent the standard bound (11) and the dual bound (18).

cannot be improved when $n = 2$. On the contrary, when $n = 3$, the dual bound provided in (18) is strictly better than the standard bound (11) for all non-negative thresholds s. Figure 4 illustrates the analogous bounds for $\mathbb{P}[\sum_{i=1}^{n} \mathbf{X}_i \geq (s, s)]$. We refer to the *dual bound on* $M_+(\mathbf{s})$ as the pointwise minimum between the two bounds provided by (20) and by the admissible choice given in (21). Note that the standard bound (12) is improved also for the sum of two risks. In the plots to follow, the *comonotonic scenario* is the case in which $\mathbb{P}[\mathbf{X}_i = \mathbf{X}_1, i \in N] = 1$.

6.1 Multivariate Value-at-Risk

An important issue for a risk manager concerning a risky position $\mathbf{X}$ is to determine the maximum aggregate loss which can occur with some given probability α. For portfolios of univariate risks, Value-at-Risk, e.g. the α-quantile of the loss distribution function, serves this purpose.

Definition 16 *For* $\alpha \in [0, 1]$*, the* Value-at-Risk *at probability level* α *for a random variable* Y *is its* α*-quantile, defined as*

$$\mathrm{VaR}_\alpha(Y) := \inf\{x \in \mathbb{R} : G(x) \geq \alpha\},$$

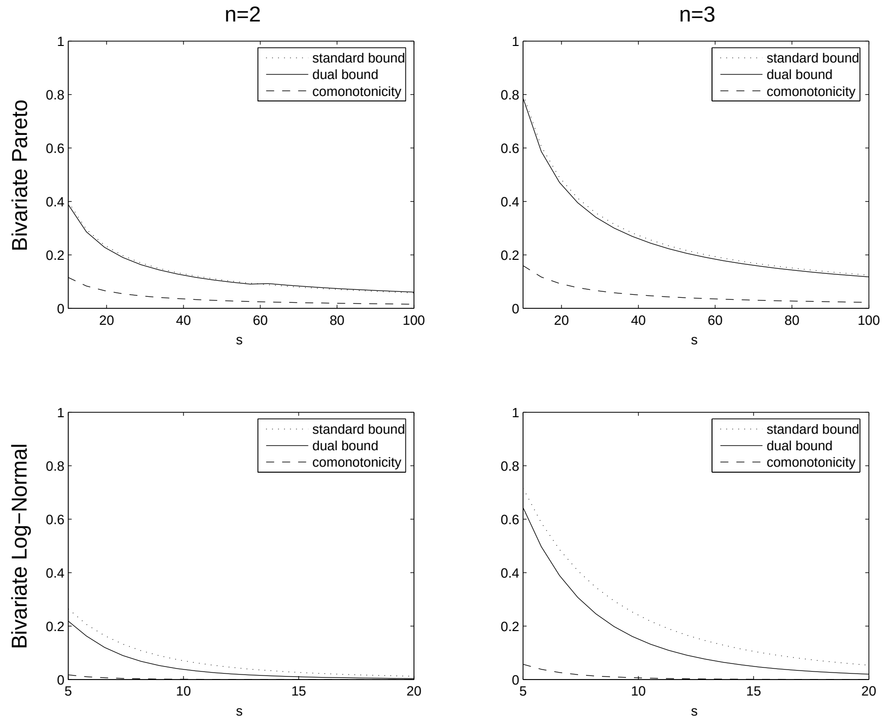

Fig. 4. Range for $\mathbb{P}[\sum_{i=1}^{n} \mathbf{X}_i \geq (s, s)]$ for two and three risks identically distributed as a bivariate Pareto or bivariate Log-Normal distribution function. Together with the comonotonic situation, we represent the standard bound (12) and the dual bound on $M_+(\mathbf{s})$.

where G is the distribution function of Y.

If G is strictly increasing, $\mathrm{VaR}_\alpha(Y)$ is the unique threshold t at which $G(t) = \alpha$. With univariate marginals, $m_\psi^{-1}(\alpha)$ is the largest $\mathrm{VaR}_\alpha(\psi(\mathbf{X}))$ over $\mathfrak{F}(F_1, \ldots, F_n)$. With multivariate marginals, Definition 16 does not make sense since, even for a continuous distribution function G, there are possibly infinitely many vectors $\mathbf{s} \in \mathbb{R}^k$ at which $G(\mathbf{s}) = \alpha$. Moreover, we may ask which events regarding $\psi(\mathbf{X})$ should be relevant for risk management.

Once the multivariate marginals of a portfolio are fixed, from a risk management viewpoint, one should be interested in bounding from above the probability that the aggregate loss amount will exceed some given threshold in all policy subgroups, i.e. $\mathbb{P}[\psi(\mathbf{X})^j \geq \mathbf{s}^j, j \in K]$. Moreover, the probability that none of the aggregate loss position for each subgroup will exceed a given threshold, i.e. $\mathbb{P}[\psi(\mathbf{X})^j < \mathbf{s}^j, j \in K]$, should be bounded from below. Problems (1) and (2) are exactly the mathematical reformulation of these two tasks.

An intuitive and immediate measure of the risk involved in a multivariate loss distribution function G is represented by its α-level sets. Considering also the α-level sets of the tail $\overline{G}$ leads to the following definition.

Definition 17 *For* $\alpha \in [0,1]$, *the* multivariate lower-orthant (LO-) Value-at-Risk at probability level α *for a increasing function* $G : \mathbb{R}^k \to \mathbb{I}$ *is the boundary of its* α*-level set, defined as*

$$\underline{\mathrm{VaR}}_\alpha(G) := \partial\{\mathbf{x} \in \mathbb{R}^k : G(\mathbf{x}) \geq \alpha\}.$$

Analogously, the multivariate upper-orthant (UO-) Value-at-Risk at probability level α *for a decreasing function* $\overline{G} : \mathbb{R}^k \to \mathbb{I}$ *is defined as*

$$\overline{\mathrm{VaR}}_\alpha(\overline{G}) := \partial\{\mathbf{x} \in \mathbb{R}^k : \overline{G}(\mathbf{x}) \leq 1 - \alpha\}.$$

If G is a distribution function, or $\overline{G}$ *is a tail function, we speak about* Value-at-Risks *for the associated random vectors.*

The α-VaRs for m_ψ and M_ψ provide conservative estimates of the α-VaRs for the aggregate loss $\psi(\mathbf{X})$. In fact, if $\mathbf{x}_1 \in \underline{\mathrm{VaR}}_\alpha(m_+)$ and $\mathbf{x}_2 \in \overline{\mathrm{VaR}}_\alpha(M_+)$, we have that

$$\mathbb{P}[\psi(\mathbf{X}) < \mathbf{s}] \geq \alpha \text{ for every } \mathbf{s} > \mathbf{x}_1,$$
$$\mathbb{P}[\psi(\mathbf{X}) \geq \mathbf{s}] \leq 1 - \alpha \text{ for every } \mathbf{s} > \mathbf{x}_2.$$

We refer to $\underline{\mathrm{VaR}}_\alpha(m_\psi)$ and $\overline{\mathrm{VaR}}_\alpha(M_\psi)$ as the *worst-possible* Value-at-Risks for the risky position $\psi(\mathbf{X})$. When it is not possible to compute m_ψ and M_ψ exactly, the α-VaRs for the corresponding dual bounds still provide conservative estimates, as stated in (5) and (6).

In Figure 5, we show worst-possible LO-VaRs for the sum of two Pareto and Log-Normal bivariate risks, while, in Figure 6, we provide UO-VaRs for the dual bound on M_ψ in case of portfolios of three risks.

An advantage of our approach is that every dual bound can be easily computed for large values of n; see Figure 7.

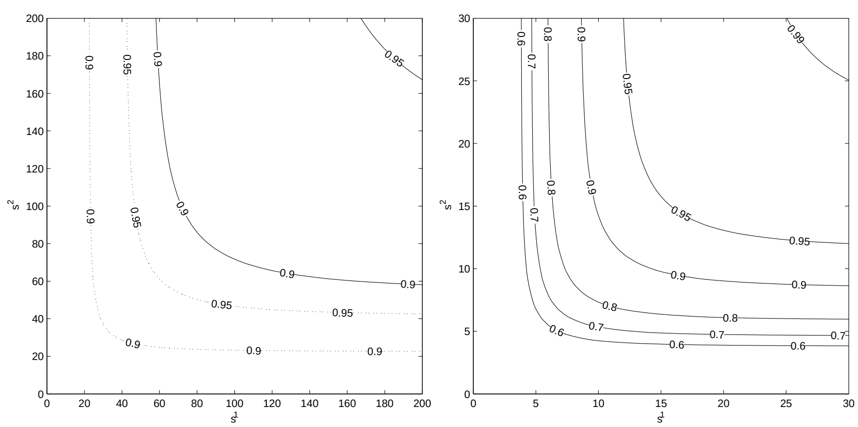

Fig. 5. Worst-possible LO-VaRs for the sum of two bivariate Pareto ($\theta = 1.2$ for the dotted line) (left) and bivariate Log-Normal (right) distributed risks.

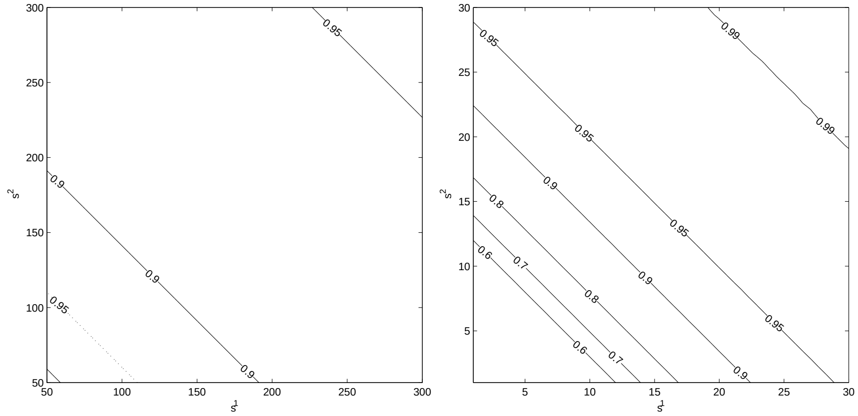

Fig. 6. UO-VaRs for the dual bound on M_ψ for the sum of three bivariate Pareto ($\theta = 1.2$ for the dotted line) (left) and bivariate Log-Normal (right) distributed risks.

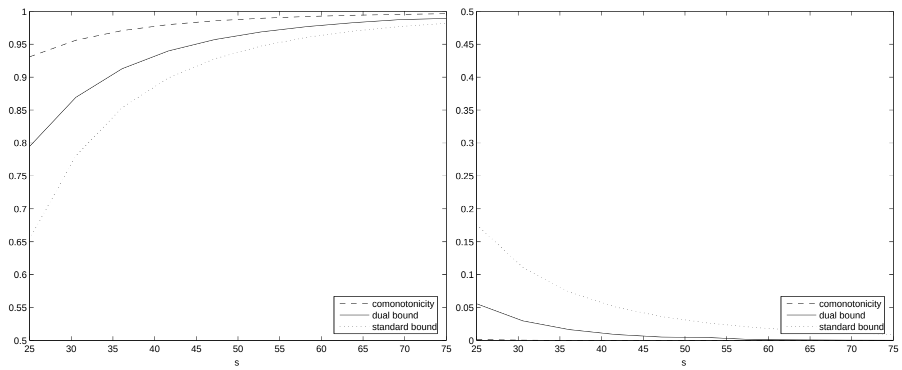

Fig. 7. Range for $\mathbb{P}[\sum_{i=1}^{5} \mathbf{X}_i < (s, s)]$ (left) and $\mathbb{P}[\sum_{i=1}^{5} \mathbf{X}_i \geq (s, s)]$ (right) for a bivariate Log-Normal portfolio.

7 Conclusions

Embrechts and Puccetti [4] propose a dual approach for the problem of determining bounds for functions of dependent risks having fixed univariate marginals. In this paper we give an extension of all results contained in the latter article to multivariate marginals. Correcting a result in Li et al. [8], we state so-called standard bounds for general functions of the underlying random vectors and give improved dual bounds for the sum of non-negative, identically distributed risks. We also derive an optimal coupling in the case of marginals which are uniformly distributed on the k-dimensional hypercube. Finally, we provide some actuarial and financial applications, including a new definition of multivariate Value-at-Risk.

Acknowledgements

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A Proof of Theorem 10

Recall from Example 7 that if $\mathbf{X}$ is uniformly distributed on $\mathbb{I}^k$ so is $(\mathbf{1}-\mathbf{X})$. Then, for $\mathbf{s} \in [k, k+1]^k$ it is possible to write:

$$\begin{aligned} M_+(\mathbf{s}) &= \sup\left\{\mathbb{P}\left[\sum_{i=1}^{k+1} \mathbf{X}_i \geq \mathbf{s}\right] : \mathbf{X}_i \backsim U(\mathbb{I}^k)\right\} \\ &= \sup\left\{\mathbb{P}\left[\sum_{i=1}^{k+1} (\mathbf{1}-\mathbf{X}_i) \leq (k+1)\mathbf{1} - \mathbf{s}\right] : \mathbf{X}_i \backsim U(\mathbb{I}^k)\right\} \\ &= \sup\left\{\mathbb{P}\left[\sum_{i=1}^{k+1} \mathbf{X}'_i \leq (k+1)\mathbf{1} - \mathbf{s}\right] : \mathbf{X}'_i \backsim U(\mathbb{I}^k)\right\}. \end{aligned}$$

Hence, to prove the theorem, it suffices to show that, for $\mathbf{s} \in (0,1]^k$,

$$\sup\left\{\mathbb{P}\left[\sum_{i=1}^{k+1} \mathbf{X}_i \leq \mathbf{s}\right] : \mathbf{X}_i \backsim U(\mathbb{I}^k)\right\} = \frac{\prod_{j=1}^{k} s^j}{k!}.$$

Define the sets $A_k := \{\mathbf{x} \in \mathbb{I}^k : \sum_{j=1}^k \frac{x^j}{s^j} \leq 1\}$, $\overline{A_k} := \mathbb{I}^k \setminus A_k$ and the function $F : \mathbb{I}^k \to \mathbb{I}^k$,

$$F(\mathbf{x}) := \begin{cases} \mathbf{x}T + \mathbf{b} & \text{if } \mathbf{x} \in A_k \\ \mathbf{x} & \text{otherwise,} \end{cases}$$

where $\mathbf{b} := (s^1, 0, \ldots, 0)$ and

$$T := \begin{pmatrix} -1 & \frac{s^2}{s^1} & 0 & \ldots & 0 \\ -\frac{s^1}{s^2} & 0 & \frac{s^3}{s^2} & \ldots & 0 \\ -\frac{s^1}{s^3} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{s^1}{s^{k-1}} & 0 & 0 & \ldots & \frac{s^k}{s^{k-1}} \\ -\frac{s^1}{s^k} & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

We further denote $F^{(0)}(\mathbf{x}) := \mathbf{x}$, $F^{(r)}(\mathbf{x}) := F \circ F^{(r-1)}(\mathbf{x}), r \geq 1$. There are several facts about F that we will need in the following.

Fact 1: $|\det(T)| = 1$. Adding to the first column of T the i-th column, $2 \le i \le k$, multiplied by $\frac{s^1}{s^i}$ and exchanging the first and the last column of the matrix so obtained, we obtain the matrix

$$T' := \operatorname{diag}\left(\frac{s^2}{s^1}, \frac{s^3}{s^2}, \dots, \frac{s^k}{s^{k-1}}, -\frac{s^1}{s^k}\right),$$

which satisfies $|\det(T)| = |\det(T')| = 1$.

Fact 2: $F(A_k) = A_k$. For every $\mathbf{x} \in A_k$ we have that

$$\begin{aligned}
\sum_{j=1}^{k} \frac{F(\mathbf{x})^j}{s^j} &= \frac{1}{s^1}\left(s^1 - \sum_{r=1}^{k} \frac{s^1}{s^r} x^r\right) + \sum_{r=2}^{k} \frac{1}{s^r}\frac{s^r}{s^{r-1}} x^{r-1} = 1 - \frac{x^k}{s^k} \le 1,\\
F(\mathbf{x})^1 &= s^1\left(1 - \sum_{r=1}^{k} \frac{x^r}{s^r}\right) \ge 0,\\
F(\mathbf{x})^j &= x^{j-1}\frac{s^j}{s^{j-1}} \ge 0, 2 \le j \le k,
\end{aligned}$$

implying that $F(A_k) \subset A_k$. Moreover, for any $\mathbf{y} \in A_k$, there is a unique vector $\mathbf{x_y} := (\mathbf{y} - \mathbf{b})T^{-1}$ with coordinates

$$\mathbf{x}_{\mathbf{y}}^j = \frac{s^j}{s^{j+1}} y^{j+1} \ge 0,\ j = 1, \dots k-1, \mathbf{x}_{\mathbf{y}}^k = s^k\left(1 - \sum_{j=1}^{k} \frac{y^j}{s^j}\right) \ge 0,$$

which satisfies $F(\mathbf{x_y}) = \mathbf{y}$ and

$$\sum_{j=1}^{k} \frac{x_{\mathbf{y}}^j}{s^j} = \sum_{j=1}^{k-1} \frac{y^{j+1}}{s^{j+1}} + 1 - \sum_{j=1}^{k} \frac{y^j}{s^j} = 1 - \frac{y^1}{s^1} \le 1,$$

implying also that $F(A_k) \supset A_k$.

Fact 3: $\sum_{\mathbf{r=0}}^{\mathbf{k}} \mathbf{F^{(r)}}(\mathbf{x}) = \mathbf{s}$, for every $\mathbf{x} \in A_k$. First, note that for $\mathbf{x} \in A_k$,

$$\begin{aligned}
F^{(j)}(\mathbf{x})^j = \frac{s^j}{s^{j-1}} F^{(j-1)}(\mathbf{x})^{j-1} &= \frac{s^j}{s^{j-1}}\frac{s^{j-1}}{s^{j-2}} F^{(j-2)}(\mathbf{x})^{j-2}\\
&= \dots = \frac{s^j}{s^1} F(\mathbf{x})^1 = s^j\left(1 - \sum_{r=1}^{k} \frac{x^r}{s^r}\right).
\end{aligned} \tag{A.1}$$

If $0 \le i < j \le k$, we have instead

$$\begin{aligned}
F^{(i)}(\mathbf{x})^j = \frac{s^j}{s^{j-1}} F^{(i-1)}(\mathbf{x})^{j-1} &= \frac{s^j}{s^{j-1}}\frac{s^{j-1}}{s^{j-2}} F^{(i-2)}(\mathbf{x})^{j-2}\\
= \dots = \frac{s^j}{s^{j-i+1}} F(\mathbf{x})^{j-i+1} &= \frac{s^j}{s^{j-i}} \mathbf{x}^{j-i}.
\end{aligned} \tag{A.2}$$

We prove now by induction that for every $i = 2, \dots, k$ we have that

$$F^{(j)}(\mathbf{x})^1 = s^1 \frac{x^{k-j+2}}{s^{k-j+2}}, 2 \le j \le i. \tag{A.3}$$

Equation (A.3) is true for $i = 2$. In fact,

$$F^{(2)}(\mathbf{x})^1 = s^1 \left(1 - \sum_{r=1}^{k} \frac{F(\mathbf{x})^r}{s^r}\right) = s^1 \left(1 - \frac{F(\mathbf{x})^1}{s^1} - \sum_{r=2}^{k} \frac{1}{s^r} \frac{s^r}{s^{r-1}} x^{r-1}\right) = s^1 \frac{x^k}{s^k}.$$

Then, assume that (A.3) holds with $i = \hat{j} < k$. Note that

$$F^{(\hat{j})}(\mathbf{x})^r = \frac{s^r}{s^{r-1}} F^{(\hat{j}-1)}(\mathbf{x})^{r-1} = \frac{s^r}{s^{r-1}} \frac{s^{r-1}}{s^{r-2}} F^{(\hat{j}-2)}(\mathbf{x})^{r-2} = \cdots = \frac{s^r}{s^1} F^{(\hat{j}-r+1)}(\mathbf{x})^1,$$

when $r = 1, \dots, \hat{j} - 1$. Since $2 \le \hat{j} - r + 1 \le \hat{j}$, and using the induction hypothesis, we conclude that

$$F^{(\hat{j})}(\mathbf{x})^r = \frac{s^r}{s^1} s^1 \frac{x^{k-\hat{j}+r-1+2}}{s^{k-\hat{j}+r-1+2}} = s^r \frac{x^{k-\hat{j}+r+1}}{s^{k-\hat{j}+r+1}}, \text{ for all } 1 \le r \le \hat{j} - 1.$$

Now we can prove (A.3) by showing that

$$\begin{aligned}
\frac{F^{(\hat{j}+1)}(\mathbf{x})^1}{s^1} &= 1 - \sum_{r=1}^{k} \frac{F^{(\hat{j})}(\mathbf{x})^r}{s^r} = \\
&= 1 - \sum_{r=1}^{\hat{j}-1} \frac{F^{(\hat{j})}(\mathbf{x})^r}{s^r} - \frac{F^{(\hat{j})}(\mathbf{x})^{\hat{j}}}{s^{\hat{j}}} - \sum_{r=\hat{j}+1}^{k} \frac{F^{(\hat{j})}(\mathbf{x})^r}{s^r} \\
&= 1 - \sum_{r=1}^{\hat{j}-1} \frac{x^{k-\hat{j}+r+1}}{s^{k-\hat{j}+r+1}} - 1 + \sum_{r=1}^{k} \frac{x^r}{s^r} - \sum_{r=\hat{j}+1}^{k} \frac{x^{r-\hat{j}}}{s^{r-\hat{j}}} = \frac{x^{k-\hat{j}+1}}{s^{k-\hat{j}+1}}.
\end{aligned}$$

Finally, we use (A.1), (A.2) and (A.3) to show that

$$\begin{aligned}
\sum_{r=0}^{k} F^{(r)}(\mathbf{x})^j &= x^j + \sum_{r=1}^{j-1} F^{(r)}(\mathbf{x})^j + F^{(j)}(\mathbf{x})^j + \sum_{r=j+1}^{k} F^{(r)}(\mathbf{x})^j \\
&= x^j + \sum_{r=1}^{j-1} s^j \frac{x^{j-r}}{s^{j-r}} + s^j \left(1 - \sum_{r=1}^{k} \frac{x^r}{s^r}\right) + \sum_{r=j+1}^{k} \frac{s^j}{s^1} F^{(r-j+1)}(\mathbf{x})^1 \\
&= x^j + \sum_{r=1}^{j-1} s^j \frac{x^{j-r}}{s^{j-r}} + s^j - \sum_{r=1}^{k} s^j \frac{x^r}{s^r} + \sum_{r=j+1}^{k} s^j \frac{x^{k-r+j+1}}{s^{k-r+j+1}} = s^j,
\end{aligned}$$

for all $j \in K$ and $\mathbf{x} \in A_k$.

Now we are ready to prove the theorem using the usual *coupling-dual* approach. Let $\mathbf{X}_1$ be uniformly distributed on $\mathbb{I}^k$ and denote with μ the corresponding measure.

$F(\mathbf{X}_1)$ is still uniformly distributed. In fact, for any fixed Borel set B in $\mathbb{I}^k$, and recalling that $F_{|A^k}$ and $F_{|\overline{A^k}}$ are one-to-one, it is true that

$$\begin{aligned} \mu^F[B] := \mu[\mathbf{x} \in \mathbb{I}^k : F(\mathbf{x}) \in B] &= \mu^F[B \cap A_k] + \mu^F[B \cap \overline{A_k}] \\ &= \mu[F^{-1}(B \cap A_k)] + \mu[F^{-1}(B \cap \overline{A_k})]. \end{aligned} \tag{A.4}$$

Since $F^{-1}(B \cap A_k) \subset A_k$ and $F^{-1}(B \cap \overline{A_k}) \subset \overline{A_k}$, (A.4) gives

$$\mu^F[B] = \mu[F^{-1}_{|A^k}(B \cap A_k)] + \mu[\mathrm{Id}(B \cap \overline{A_k}] = \mu[F^{-1}_{|A^k}(B \cap A_k)] + \mu[B \cap \overline{A_k}]. \tag{A.5}$$

$F^{-1}_{|A_k}(\mathbf{y})$ is an affine transformation of the form $\mathbf{y}T^{-1} - \mathbf{b}T^{-1}$ with $|\det(T^{-1})| = |(\det T)^{-1}| = 1$. By Billingsley [1, pp.172–173] we have that $\mu[F^{-1}_{|A^k}(B \cap A_k)] = \mu[B \cap A_k]$ and hence, from (A.5), that $\mu^F[B] = \mu[(B \cap A_k)] + \mu[B \cap \overline{A_k}] = \mu[B]$. We conclude that $F^{(r)}(\mathbf{X}_1) \backsim U(\mathbb{I}^k)$ for all $r \geq 1$. Therefore, we can define the following coupling:

$$\mathbf{X}_i^C := F^{(i-1)}(\mathbf{X}_1), i = 1, \ldots, k+1.$$

Since $\sum_{\mathbf{r=0}}^{\mathbf{k}} \mathbf{F}^{(\mathbf{r})}(\mathbf{x}) = \mathbf{s}$ for every $\mathbf{x} \in A_k$, we get, for all $\mathbf{s} \in \mathbb{I}^k$,

$$\mathbb{P}\left[\sum_{i=1}^{k+1} \mathbf{X}_i^C \leq \mathbf{s}\right] \geq \mathbb{P}[\mathbf{X}_1^C \in A_k] = \frac{\prod_{j=1}^k s^j}{k!}.$$

We prove the opposite inequality by finding an admissible choice yielding the same value for the corresponding dual problem:

$$\begin{aligned} &\sup\Big\{\mathbb{P}\Big[\sum_{i=1}^{k+1} \mathbf{X}_i \leq \mathbf{s}\Big] : \mathbf{X}_i \backsim U(\mathbb{I}^k), 1 \leq i \leq k+1\Big\} \\ &= \inf\Big\{(k+1)\int_{\mathbb{I}^k} f dU(\mathbb{I}^k) : f \in L^1(U(\mathbb{I}^k)) \text{ with} \\ &\qquad \sum_{i=1}^{k+1} f_i(\mathbf{x}_i) \geq 1_{(-\infty, \mathbf{s}]}\left(\sum_{i=1}^{k+1} \mathbf{x}_i\right) \text{ for all } \mathbf{x}_i \in \mathbb{I}^k, 1 \leq i \leq k+1\Big\}. \end{aligned} \tag{A.6}$$

As dual choice we choose the function $f : \mathbb{I}^k \to \mathbb{R} : f(\mathbf{x}) = [1 - \sum_{j=1}^k \frac{x^j}{s^j}]^+$. Since $f \geq 0$, it is sufficient to fix an arbitrary $\mathbf{x} \in \mathbb{I}^k$ such that $\sum_{i=1}^{k+1} x_i^j \leq s^j$, $j \in K$, and show that $\sum_{i=1}^{k+1} f(\mathbf{x}_i) \geq 1$. Define the sets $I := \{i \in N : \sum_{j=1}^k \frac{x_i^j}{s^j} \leq 1\}, \overline{I} := N \setminus I$. As $\sum_{i=1}^{k+1} \frac{x_i^j}{s^j} \leq 1$, $j \in K$,we have that

$$k \geq \sum_{j=1}^k \sum_{i=1}^{k+1} \frac{x_i^j}{s^j} = \sum_{i=1}^{k+1} \sum_{j=1}^k \frac{x_i^j}{s^j} = \sum_{i \in I} \sum_{j=1}^k \frac{x_i^j}{s^j} + \sum_{i \in \overline{I}} \sum_{j=1}^k \frac{x_i^j}{s^j}.$$

Since $\sum_{i\in\bar{I}}\sum_{j=1}^{k}\frac{x_i^j}{s^j} > \left|\bar{I}\right|$, the latter yields $\sum_{i\in I}\sum_{j=1}^{k}\frac{x_i^j}{s^j} \leq k-\left|\bar{I}\right|$. Finally, we can write

$$\sum_{i=1}^{k+1} f(\mathbf{x}_i) = \sum_{i\in I} f(\mathbf{x}_i) + \sum_{i\in\bar{I}} f(\mathbf{x}_i) = \sum_{i\in I} f(\mathbf{x}_i)$$
$$= |I| - \sum_{i\in I}\sum_{j=1}^{k}\frac{x^j}{s^j} \geq |I| - (k-\left|\bar{I}\right|) = k+1-k = 1,$$

which gives admissibility of f. Substituting f in (A.6), we obtain

$$\sup\left\{\mathbb{P}\left[\sum_{i=1}^{k+1}\mathbf{X}_i \leq \mathbf{s}\right] : \mathbf{X}_i \backsim U(\mathbb{I}^k), 1 \leq i \leq k+1\right\}$$
$$\leq (k+1)\int_{\mathbb{I}^k}\left[1-\sum_{j=1}^{k}\frac{x^j}{s^j}\right]^+ \otimes_{j=1}^{k}(dx^j)$$
$$= (k+1)\int_0^{s^1}\int_0^{s^2(1-\frac{x^1}{s^1})}\ldots\int_0^{s^k(1-\sum_{j=1}^{k-1}\frac{x^j}{s^j})}\left(1-\sum_{j=1}^{k}\frac{x^j}{s^j}\right)\otimes_{j=1}^{k}(dx^j)$$
$$= (k+1)\int_0^{s^1}\int_0^{s^2(1-\frac{x^1}{s^1})}\ldots$$
$$\int_0^{s^{k-1}(1-\sum_{j=1}^{k-2}\frac{x^j}{s^j})} -\frac{s^k\left(1-\sum_{j=1}^{k}\frac{x^j}{s^j}\right)^2}{2}\Bigg|_0^{s^k(1-\sum_{j=1}^{k-1}\frac{x^j}{s^j})} \otimes_{j=1}^{k-1}(dx^j)$$
$$= \frac{(k+1)s^k}{2}\int_0^{s^1}\int_0^{s^2(1-\frac{x^1}{s^1})}\ldots\int_0^{s^{k-1}(1-\sum_{j=1}^{k-2}\frac{x^j}{s^j})}\left(1-\sum_{j=1}^{k-1}\frac{x^j}{s^j}\right)^2 \otimes_{j=1}^{k-1}(dx^j)$$
$$= \cdots = \frac{\prod_{j=1}^{k}s^j}{k!},$$

which concludes the proof. It is easy to show that the function $f(\mathbf{x}) := [1-\frac{n-1}{k}\sum_{j=1}^{k}\frac{x^j}{s^j}]^+$ is a dual choice for (A.6) for all $\mathbf{s} \in [\mathbf{0},+\infty), k \geq 2$ and $n \geq 2$. Hence an upper bound on $M_+(\mathbf{s})$ is always available; see Remark 11 (ii).

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